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THE WEAK NON-LINEAR FLUCTUATIONS IN THE RADIUS OF A CONDENSED DROP IN AN ACOUSTIC FIELD*

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The non-linear effects of heat and mass transfer of a drop in a vapour medium when there is a periodic change in the pressure brought about by an acoustic wave, the wavelength of which is considerably greater than the radius of the drop, are investigated. The asymptotic equations which describe the behaviour of the mean radius of the drop in an approximation which is quadratic in the field amplitude are obtained using the multiple scale method /2/ on the basis of the closed system of equations, which describes spherically symmetric processes around a single drop /1/.

When investigating the propgation of sound in mixture of a vapour with droplets, it is assumed in the linear formulation /3/ that the radii of the drops pulsate around stationary positions as a result of periodic phase changes. However, as a consequence of the manifestation of non-linear heat and mass transfer effects, the amounts of the fluid which have vaporised and condensed during a single vibrational period will, in general, be different (the "rectified heat transfer effect" /4/). After a time equal to a large number of vibrational periods, the mean radius of the drop may have appreciable changed and, correspondingly, there may also be changes in such important parameters characterizing the weight as a whole as the mass content of the disperse phase and the size distribution function for the drops.

1. *Formulation of the problem.* A spherical drop of radius a is placed in an unbounded space occupied by vapour, which is considered within the framework of the model as an ideal gas. In the unperturbed state, the vapour and the liquid are in equilibrium, that is, $p_* = p_s(T_*)$, where p and T are the pressure and temperature, parameters on the saturation line are given the index s and the parameters for the unperturbed state are given an asterisk. When acoustic waves of wavelength $L \gg a$ $(L = 2\pi C_g/\omega$, where C_g is the velocity of sound in the vapour and ω is the angular frequency) act on this system, it may be assumed that, in a coordinate system associated with the centre of the drop and in a domain $r \ll L$, where r is the radial coordinate measured from the centre of the drop, the temperature and velocity distributions in the fluid and in the vapour will be spherically symmetric, and that the pressure will be solely dependent on the time $t: p = p(t)$ (homobaricity /1/). In this case the closed system of equations and boundary conditions which describe the heat and mass transfer between an incompressible drop and its vapour in a field of variable pressure has the form /1/

$$\begin{gathered}
p_g(t) = R_g T_g(\eta, t)\rho_g(\eta, t), \quad \rho_l = \text{const} \\
\rho_l c_l \left[a^2 \frac{\partial T_l}{\partial T} - a \frac{da}{dt}\eta \frac{\partial T_l}{\partial \eta}\right] = \lambda_l \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial T_l}{\partial \eta}\right) \\
\rho_g c_g \left[a^2 \frac{\partial T_g}{\partial t} + a\left(w_g - \eta \frac{da}{dt}\right)\frac{\partial T_g}{\partial \eta}\right] = \lambda_g \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial T_g}{\partial \eta}\right) + a^2 \frac{dp_g}{dt} \\
w_g = \frac{w_{ga}}{\eta^2} + \frac{\gamma - 1}{\gamma p_g a}\left(\lambda_g \frac{\partial T_g}{\partial \eta} + \frac{Q_g}{\eta^2}\right) - \frac{a}{3\gamma p_g}\left(\eta - \frac{1}{\eta^2}\right)\frac{dp_g}{dt} \\
T_\alpha|_{\eta=1} = T_a, \; Q_\alpha = -\lambda_\alpha \frac{\partial T_\alpha}{\partial \eta}\bigg|_{\eta=1}, \quad (\alpha = g, l), \quad \frac{\partial T_l}{\partial \eta}\bigg|_{\eta=0} = \frac{\partial T_g}{\partial \eta}\bigg|_{\eta=\infty} = 0 \\
w_{la} = \frac{da}{dt} + \frac{\xi}{\rho_l} \equiv 0, \quad w_{ga} = \frac{da}{dt} + \frac{\xi}{\rho_{ga}}, \quad Q_g - Q_l = -\xi l a \\
\xi = \frac{\beta[p_s(T_a) - p_g]}{\sqrt{2\pi R_g T_a}}, \quad p_s(T) = p_* \exp\left[\frac{1}{k_s}\left(1 - \frac{T_*}{T}\right)\right], \quad k_s = \frac{R_g T_*}{l}
\end{gathered} \tag{1.1}$$

Here, ρ, w, Q and ξ are the density, the radial velocity, the reduced thermal flux on the interphase boundary and the intensity of vaporization per unit area of surface, λ, c, γ, R_g and l (quantities which are assumed to be constants) are the thermal conductivity, the heat capacity (in the case of the gas, this is at constant pressure), the adiabatic index of

*Prikl. Matem. Mekhan., 53, 2, 261-270, 1989 the gas, the gas constant and the latent heat of vaporization, and $\beta(p_g, T_a)$ is the coefficient of accommodation, the dependence of which on temperature and pressure is assumed to be known. The indices g and l refer respectively to the gas and the liquid while the parameters on the interphase boundary are indicated by the index a. In writing down (1.1), use was made of the space-time transformation $(r,t) \to (\eta, t)$, where $\eta = r/a(t)$. This enables one to reduce the problems from domains with a variable boundary $r = a(t)$ to problems with a fixed boundary.

We shall subsequently study the solutions of system (1.1) when the changes in the pressure around the value p_* are governed by periodic laws

$$p_g = p_* (1 + \varepsilon\varphi(t)), \ \varphi(t) = \varphi(t + 2\pi/\omega), \ |\varphi| \leqslant 1 \tag{1.2}$$

where $\varphi(t)$ is a periodic function which can be represented in the form of a Fourier series. The relative amplitude of the pressure perturbation is taken as the small parameter of the problem: $0 < \varepsilon \ll 1$.

2. Method of solution. The radial velocity and the density of the gas can be eliminated from system (1.1) and the remaining unknown dependent variables are sought in the form of the asymptotic series

$$\begin{gathered}
T_a = T_* (\theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots), \quad T_\alpha = T_* (u_0^{(\alpha)} + \varepsilon u_1^{(\alpha)} + \varepsilon^2 u_2^{(\alpha)} + \ldots) \\
a = a_0 (1 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots), \quad Q_\alpha = \lambda_\alpha T_* (q_0^{(\alpha)} + \varepsilon q_1^{(\alpha)} + \varepsilon^2 q_2^{(\alpha)} + \ldots) \\
\xi = \frac{\rho_{g*}\varkappa_g}{a_0} (j_0 + \varepsilon j_1 + \varepsilon^2 j_2 + \ldots) \quad \left(\varkappa_\alpha = \frac{\lambda_\alpha}{\rho_{\alpha*} c_\alpha}, \ \rho_{g*} = \frac{p_*}{R_g T_*}, \ \alpha = g, l\right)
\end{gathered} \tag{2.1}$$

Use was made of the method of multiple scales /2/ in constructing uniformly good expansions. According to this method the functional dependence of the unknowns $a_m, u_m^{(l)}, u_m^{(g)}$ and so on $(m = 0, 1, 2, \ldots)$ on t is treated as a dependence on a set of times $\{t_k\}$, $t_k = \varepsilon^k t$, $k = 0, 1, 2, \ldots$, that is, the operator for differentiation with respect to t is expanded in the asymptotic series

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots \tag{2.2}$$

The function $\varphi(t)$ occurring in (12) must be considered as a function solely of the "fast" time t_0.

By substituting (1.2), (2.1) and (2.2) into (1.1) and collecting the terms in the same powers of ε, it can be seen that $\theta_0 \equiv 1$, $u_0^{(\alpha)} \equiv 1$, $q_0^{(\alpha)} \equiv 0$ $(\alpha = l, g)$, $j_0 \equiv 0$, $a_0 = a_0(t_1, t_2, \ldots)$ and the linear inhomogeneous system

$$\begin{gathered}
\frac{a_0^2}{\varkappa_g} \frac{\partial u_m^{(l)}}{\partial t_0} - \frac{k_\varkappa}{\eta^2} \frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial u_m^{(l)}}{\partial \eta}\right) = f_m^{(l)}, \quad u_m^{(l)}\big|_{\eta=1} = \theta_m, \quad \frac{\partial u_m^{(l)}}{\partial \eta}\bigg|_{\eta=0} = 0 \\
\frac{a_0^2}{\varkappa_g} \frac{\partial u_m^{(g)}}{\partial t_0} - \frac{1}{\eta^2} \frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial u_m^{(g)}}{\partial \eta}\right) = \\
k_\gamma \frac{a_0^2}{\varkappa_g} \frac{d\varphi}{dt_0} \delta_{m1} + f_m^{(g)}, \quad u_m^{(g)}\big|_{\eta=1} = \theta_m, \quad \frac{\partial u_m^{(g)}}{\partial \eta}\bigg|_{\eta=\infty} = 0 \\
\frac{a_0^2}{\varkappa_g} \frac{\partial a_m}{\partial t_0} + k_\rho j_m = f_m^{(a)}, \quad j_m - \frac{a_0}{d_\sigma}\theta_m = -\frac{k_s a_0}{d_\sigma}\varphi\delta_{m1} + f_m^{(j)}
\end{gathered} \tag{2.3}$$

$$\begin{gathered}
q_m^{(\alpha)} = -\frac{\partial u_m^{(\alpha)}}{\partial \eta}\bigg|_{\eta=1}, \quad \alpha = g, l, \quad \frac{k_s}{k_\gamma}(q_m^{(g)} - k_\lambda q_m^{(l)}) + j_m = f_m^{(q)} \\
k_\gamma = 1 - 1/\gamma, \quad k_\rho = \rho_{g*}/\rho_l, \quad k_\lambda = \lambda_l/\lambda_g, \quad k_\varkappa = \varkappa_l/\varkappa_g \\
d_\sigma = \varkappa_g \sqrt{2\pi R_g T_*}/(\beta_* l), \quad \beta_* = \beta(p_*, T_*)
\end{gathered}$$

is obtained for determining the m-th $(m > 0)$ approximation.

Here, δ_{mk} is the Kronecker delta and $f_m^{(\alpha)}$ are functions which are determined from approximations lower than the m-th approximation $(\alpha = l, g, a, j, q)$.

By t_0-periodic functions, we shall henceforth understand functions which are periodic with respect to the fact time t_0 and are representable in the form of a corresponding Fourier series in t_0

$$x(t_0, t_1, t_2, \ldots) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} x_n^\circ (t_1, t_2, \ldots) e^{in\omega t_0}\right\} \tag{2.4}$$

where x_n° are complex-valued functions (complex amplitudes).

The search for solutions of (2.3) in the class of t_0-periodic functions leads to systems of the type (2.3) for the complex amplitudes where the operator $\partial/\partial t_0$ passes into $i\omega n$. In particular, the complex amplitudes of the temperature distributions in the liquid and in the gas satisfy the equations

$$s_n u_{mn}^{(l)\circ} - \frac{k_\varkappa}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2 \frac{\partial u_{mn}^{(l)\circ}}{\partial\eta}\right) = f_{mn}^{(l)\circ},\quad u_{mn}^{(l)\circ}|_{\eta=1} = \overset{\circ}{\theta}_{mn},\quad \frac{\partial u_{mn}^{(l)\circ}}{\partial\eta}\bigg|_{\eta=0} = 0 \tag{2.5}$$
$$s_n u_{mn}^{(g)\circ} - \frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2 \frac{\partial u_{mn}^{(g)\circ}}{\partial\eta}\right) = k_\gamma s_n \varphi_n{}^\circ\delta_{m1} + f_{mn}^{(g)\circ},\quad u_{mn}^{(g)\circ}|_{\eta=1} = \overset{\circ}{\theta}_{mn}$$
$$\frac{\partial u_{mn}^{(g)\circ}}{\partial\eta}\bigg|_{\eta=\infty} = 0\quad s_n = s_n(t_1, t_2, \ldots) = i\omega n a_0^2/\varkappa_g$$

If, for any $\operatorname{Re}\{\sqrt{s_n}\} \geqslant 0$, the modulus of the function $f_{mn}^{(g)\circ}(\eta, t_1, t_2, \ldots)$ decreases to a sufficient extent as $\eta\to\infty$ $(|f_{mn}^{(g)\circ}| = o(\eta^{-2}))$, the solutions of the inhomogeneous problems (2.5) are representable in the form

$$u_{mn}^{(l)\circ} = \eta^{-1}[A_{mn}S_n(\eta) + G_{mn}^{(l)}(\eta)],\quad A_{mn} = (\overset{\circ}{\theta}_{mn} - G_{mn}^{(l)}(1))/S_n(1)$$
$$u_{mn}^{(g)\circ} = k_\gamma\varphi_n{}^\circ\delta_{m1} + \eta^{-1}[B_{mn}e_n^-(\eta) + G_{mn}^{(g)}(\eta)],\quad B_{mn} = \overset{\circ}{\theta}_{mn} - G_{mn}^{(g)}(1) - k_\gamma\varphi_n{}^\circ\delta_{m1} \tag{2.6}$$
$$G_{mn}^{(l)}(\eta) = (k_\varkappa s_n)^{-1/2}\int_0^\eta [C_n(\eta)S_n(x) - S_n(\eta)C_n(x)]\, x f_{mn}^{(l)\circ}(x)\,dx$$
$$G_{mn}^{(g)}(\eta) = \frac{1}{2}s_n^{-1/2}\left[e_n^-(\eta)\int_1^\eta e_n^+(x)\, x f_{mn}^{(g)\circ}(x)\,dx + e_n^+(\eta)\int_\eta^\infty e_n^-(x)\, x f_{mn}^{(g)\circ}(x)\,dx\right]$$
$$S_n(\eta) = \operatorname{sh}\left(\eta\sqrt{\frac{s_n}{k_\varkappa}}\right),\quad C_n(\eta) = \operatorname{ch}\left(\eta\sqrt{\frac{s_n}{k_\varkappa}}\right),$$
$$e_n^\pm(\eta) = e^{\pm(\eta-1)\sqrt{s_n}},\quad \operatorname{Re}\{\sqrt{s_n}\} \geqslant 0$$

From this, it is possible to find the complex amplitudes of the thermal fluxes

$$q_{mn}^{(l)\circ} = -\frac{\partial u_{mn}^{(l)\circ}}{\partial\eta}\bigg|_{\eta=1} = -h_n^{(l)}\overset{\circ}{\theta}_{mn} + \frac{1}{k_\varkappa}F_{mn}^{(l)},\quad h_n^{(l)} = \sqrt{\frac{s_n}{k_\varkappa}}\operatorname{cth}\sqrt{\frac{s_n}{k_\varkappa}} - 1$$
$$q_{mn}^{(g)\circ} = -\frac{\partial u_{mn}^{(g)\circ}}{\partial\eta}\bigg|_{\eta=1} = h_n^{(g)}(\overset{\circ}{\theta}_{mn} - k_\gamma\varphi_n{}^\circ\delta_{m1}) - F_{mn}^{(g)},\quad h_n^{(g)} = 1 + \sqrt{s_n} \tag{2.7}$$
$$F_{mn}^{(l)} = [S_n(1)]^{-1}\int_0^1 \eta f_{mn}^{(l)\circ}S_n(\eta)\,d\eta,\quad F_{mn}^{(g)} = \int_1^\infty \eta f_{mn}^{(g)\circ}e_n^-(\eta)\,d\eta$$

It therefore follows from (2.3), (2.4) and (2.7) that, in the m-th approximation, the unknown complex amplitudes $\overset{\circ}{a}_{mn}, \overset{\circ}{\theta}_{mn}, \overset{\circ}{j}_{mn}$ satisfy the linear inhomogeneous algebraic equation

$$L_n\mathbf{X}_{mn} = \mathbf{Y}_{mn} \tag{2.8}$$
$$L_n = \begin{Vmatrix} s_n & 0 & k_\rho \\ 0 & -a_0 d_\sigma^{-1} & 1 \\ 0 & k_s k_\gamma^{-1}[h_n^{(g)} + k_\lambda h_n^{(l)}] & 1 \end{Vmatrix},\quad \mathbf{X}_{mn} = \begin{Vmatrix} \overset{\circ}{a}_{mn} \\ \overset{\circ}{\theta}_{mn} \\ \overset{\circ}{j}_{mn} \end{Vmatrix},\quad \mathbf{Y}_{mn} = \begin{Vmatrix} y_{mn}^1 \\ y_{mn}^2 \\ y_{mn}^3 \end{Vmatrix}$$
$$y_{mn}^1 = f_{mn}^{(a)\circ},\quad y_{mn}^2 = -k_s a_0 d_\sigma^{-1}\varphi_n{}^\circ\delta_{m1} + f_{mn}^{(j)\circ}$$
$$y_{mn}^3 = k_s\varphi_n{}^\circ h_n^{(g)}\delta_{m1} + k_s k_\gamma^{-1}[F_{mn}^{(g)} + k_\lambda k_\varkappa^{-1}F_{mn}^{(l)}] + f_{mn}^{(q)\circ}$$

Let $\omega \neq 0$, then, when $n \neq 0$, it is obvious that $\det L_n \neq 0$ and a solution $\mathbf{X}_{mn}$ exists and it is unique. When $n = 0$, the matrix of the system degenerates and the $\operatorname{rank} L_0 = 2$. In this case, the condition $\operatorname{rank} M_0 = 2$, where M_0 is the extended matrix of the system when $n = 0$ will be a necessary and sufficient condition for the solution to exist. This condition can be written in the expanded form

$$\frac{1}{k_\rho}\left(\frac{k_\gamma}{k_s} + \frac{d_\sigma}{a_0}\right) f_{m0}^{(a)\circ} = (k_\gamma - k_s)\varphi_0{}^\circ\delta_{m1} + \frac{d_\sigma}{a_0}f_{m0}^{(j)\circ} + \frac{k_\gamma}{k_s}f_{m0}^{(q)\circ} + F_{m0}^{(g)} + \frac{k_\lambda}{k_\varkappa}F_{m0}^{(l)} \tag{2.9}$$

$$F_{m0}^{(g)} = \lim_{s_n\to 0} F_{mn}^{(g)} = \int_0^\infty \eta f_{m0}^{(g)\circ}\, d\eta, \quad F_{m0}^{(l)} = \lim_{s_n\to 0} F_{mn}^{(l)} = \int_0^1 \eta^2 f_{m0}^{(l)\circ}\, d\eta$$

Thus, it has been shown that the solution of system (2.3) in the class of t_0-periodic functions only exists if condition (2.9) is satisfied which actually "controls" the behaviour of the mean values. If condition (2.9) is satisfied, then the solution is unique with an accuracy up to the solution of the homogeneous system (2.3) which has the form

$$a_m = a_m(t_1, t_2, \ldots), \quad \theta_m \equiv j_m \equiv q_m^{(l)} = q_m^{(g)} \equiv 0, \quad u_m^{(l)} \equiv 0, \; u_m^{(g)} \equiv 0$$

3. *The first approximation. Stationary pulsations.* It is seen from (1.1), (1.2) and (2.1)-(2.3) that, to a first approximation $(m = 1)$, the functions $f_1^{(\alpha)}$ and their complex amplitudes $f_{1n}^{(\alpha)\circ}$ have the form

$$f_1^{(\alpha)} \equiv f_{1n}^{(\alpha)\circ} \equiv 0, \quad \alpha = j, q, g, l; \quad n = 0, 1, 2, \ldots; \quad f_1^{(a)} \equiv f_{10}^{(a)\circ} = -a_0 \varkappa_g^{-1} \partial a_0/\partial t_1 \tag{3.1}$$

The condition for the existence of t_0-periodic solutions when $m = 1$ yields

$$(a_0 + k_s k_\gamma^{-1} d_\sigma)\, \partial a_0/\partial t_1 = -k_\rho k_s k_\gamma^{-1} (k_\gamma - k_s) \varkappa_g \varphi_0^\circ \tag{3.2}$$

This equation is integrated:

$$^1/_2 a_0^2 + k_s k_\gamma^{-1} d_\sigma a_0 = -k_\rho k_s k_\gamma^{-1} (k_\gamma - k_s) \varphi_0^\circ \varkappa_g t_1 + C(t_2, t_3, \ldots) \tag{3.3}$$

It is immediately seen from this that, when $(k_\gamma - k_s)\varphi_0^\circ \neq 0$, the mean radius of the drop will change in the scale of the "slow" time t_1. It should be mentioned that, when $\varphi_0^\circ \neq 0$, fluctuations in the pressure do not occur around the mean value p_* but around $p_*(1 + \varepsilon\varphi_0^\circ)$ or, in other words, the system is not at equilibrium form the outset but weakly perturbed about this position. The quantity $k_\gamma - k_s$ represents the difference between the angular coefficients of the tangents to the adiabatic curve and the saturation line at the point (T_*, p_*) in the (T, p) plane. The sign of this quantity determines what the signs of the difference between the temperature in the gas remote from the drop and the saturation temperature, to which the temperature of the surface of the drop tends, will be.

Now let condition (3.2) be satisfied. Then, t_0-periodic solutions exist for system (3.2) when $m = 1$.

When $n \neq 0$, it follows from (3.1) that the components of the inhomogeneity vector Y_{1n} in (2.8) have the form $y_{1n}^1 = 0$, $y_{1n}^2 = -k_s a_0 \varphi_n^\circ d_\sigma^{-1}$, $y_{1n}^3 = k_s \varphi_n^\circ h_n^{(g)}$. Whence, by using Cramer's rule, the complex amplitudes a_{1n}°, θ_{1n}° and j_{1n}° can be found.

$$\overset{\circ}{\alpha}_{1n} = -k_s \varphi_n^\circ \Delta_n^{(\alpha)}/\Delta_n, \quad \alpha = \theta, j, a; \quad \Delta_n = \det L_n \tag{3.4}$$
$$\Delta_n^{(j)} = s_n a_0 k_\gamma^{-1} d_\sigma^{-1} [(k_\gamma - k_s) h_n^{(g)} - k_s k_\lambda^{*} h_n^{(l)}], \quad \Delta_n^{(a)} = -k_\rho s_n^{-1} \Delta_n^{(j)}$$
$$\Delta_n^{(\theta)} = s_n (a_0 d_\sigma^{-1} + h_n^{(g)}), \quad \Delta_n = d_\sigma a_0^{-1} \Delta_n^{(j)} - \Delta_n^{(\theta)}$$

The dependences of the phase $\psi = \arg a_{11}^\circ$ and the reduced amplitude $A = k_\rho^{-1} |a_{11}^\circ|$ of the fluctuations in the radius of a water drop in water vapour at a pressure $p_* = 0.1$ MPa on the frequency and mean radius are illustrated in Fig.1. The solid lines correspond to calculations with an accommodation coefficient $\beta_* = 0.04$. The dashed lines were calculated using a quasi-equailibrium phase transition scheme (this scheme is realized, if β formally tends to infinity and $T_a = T_s(p_g)$ in the case of a quasi-equilibrium phase transition scheme /1, 4/). In this case the dependence of the phase and amplitude of the fluctuations on a_0 and ω occurs in (3.4) in the form of a dependence on a single selfsimilar parameter $\omega a_0^2/\varkappa_g$. Curves *1-4* correspond to a frequency ν, equal to 1, 10, 100, and 1000 kHz. When the frequency is reduced the solid curves tend to move towards the dashed curves which is indicative of the greater "equilibrium nature" of the phase transition at low frequencies as compared with that at high frequencies.

At larger radii a_0 and a fixed frequency ω, the value of the argument of ψ, as follows from (3.4), tends to its limiting value

$$\psi_\infty = -\operatorname{arctg}(1 + K\omega_\sigma^{1/2});$$
$$\omega_\sigma = 2\omega d_\sigma^2/\varkappa_g, \quad K = k_\gamma^{-1} k_s (1 + k_\lambda k_\varkappa^{-1/2})$$

and $-\pi/4 \leqslant \psi_\infty \leqslant -\pi/2$. In the region of low values of $\omega a_0^2/\varkappa_g$, the amplitude A increases in inverse proportion to $\omega a_0^2 (A \to k_s (k_\gamma - k_s) [(\omega a_0^2/\varkappa_g)(1 + k_s d_\sigma/a_0)]^{-1})$ and $\psi \to \pi/2$. It should be noted in this connection that, in order for the reasoning to be correct, it is necessary to assume that the constraint on the frequencies, amplitudes and radii $(\omega a_0^2/\varkappa_g)(1 + k_s d_\sigma/a_0) \gg \varepsilon k_s(k_\gamma - k_s)$ is satisfied. Otherwise, the term εa_1 in the corresponding expansion (2.1) cannot be considered as being small compared with unity.

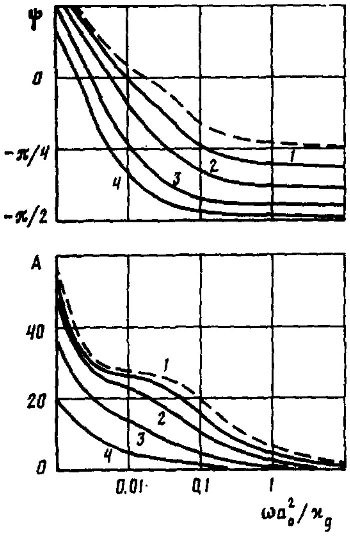

Fig.1

The expressions for the complex amplitudes of the temperature distributions and the heat flux on the interphase boundary from the gas side have the form

$$
\begin{gathered}
\overset{\circ}{u}{}_{1n}^{(l)} = \overset{\circ}{\theta}_{1n} y_n(\eta), \quad \overset{\circ}{u}{}_{1n}^{(g)} = k_\gamma \varphi_n{}^\circ + (\overset{\circ}{\theta}_{1n} - k_\gamma \varphi_n{}^\circ) z_n(\eta) \\
\overset{\circ}{q}{}_{1n}^{(g)} = h_n^{(g)} (\overset{\circ}{\theta}_{1n} - k_\gamma \varphi_n{}^\circ), \quad z_n(\eta) = \eta^{-1} e_n^-(\eta), \quad y_n(\eta) = \eta^{-1} S_n(\eta)/S_n(1)
\end{gathered} \tag{3.5}
$$

according to (2.6), (2.7) and (3.1).

When $n = 0$, the determinant of the system is equal to zero. From Eq.(2.8) and condition (3.2) which ensures the compatibility of the equations, a solution is found with an accuracy up to the solution of the homogeneous system

$$
\overset{\circ}{\theta}_{10} = k_s \varphi_0{}^\circ \frac{k_\gamma (a_0 + d_\sigma)}{a_0 + k_s d_\sigma}, \quad \overset{\circ}{j}_{10} = k_s \varphi_0{}^\circ \frac{(k_\gamma - k_s) a_0}{a_0 + k_s d_\sigma}, \quad \overset{\circ}{a}_{10} = \overset{\circ}{a}_{10}(t_1, t_2, \ldots) \tag{3.6}
$$

Eqs.(3.4)-(3.6) determine all the harmonics of the required functions when $m = 1$. The problem of the pulsations of a drop may therefore be considered as being solving in the first approximation.

However, if the system is at equilibrium $(\varphi_0{}^\circ = 0)$ at the beginning then, according to (3.2), in the first approximation the radius of the drop will execute stationary fluctations $(\partial a_0/\partial t_1 = 0)$ and it is not possible to determine the change in the mean radius of the drop from an analysis of the first approximation (linear analysis) since it will change on a scale which is "slower" than that of t_1 (the non-linear effect).

4. The equation for the dynamics of the mean radius of the drop. It is subsequently assumed that $\varphi_0{}^\circ = 0$. In this case all the required functions in the first approximation, apart from, perhaps, $a_{10}{}^\circ$, are independent of t_1. However, in the case of the given problem, one should put $a_{10}{}^\circ = 0$ since the meaning of a mean radius is embedded in a_0. From this and from (1.1), (1.2) and (2.1)-(2.3), one finds the inhomogeneities $f_2^{(\alpha)}$ $(\alpha = a, j, q, g, l)$ in the second approximation $(m = 2)$

$$
\begin{gathered}
f_2^{(a)} = -\frac{a_0}{\varkappa_g} \frac{\partial a_0}{\partial t_2}, \quad f_2^{(q)} = -a_1 j_1 \\
f_2^{(j)} = \frac{a_0}{d_\sigma} \left\{ \left(\frac{1}{2k_s} + \beta_T - \frac{3}{2} \right) \theta_1{}^2 + \left[\beta_p - k_s \left(\beta_T - \frac{1}{2} \right) \right] \varphi \theta_1 - k_s \beta_p \varphi^2 \right\} \\
f_2^{(l)} = \frac{a_0{}^2}{\varkappa_g} \left(\frac{\partial a_1}{\partial t_0} \eta \frac{\partial u_1^{(l)}}{\partial \eta} - 2a_1 \frac{\partial u_1^{(l)}}{\partial t_0} \right)
\end{gathered} \tag{4.1}
$$

$$f_2^{(g)} = \frac{u_1^{(g)}}{\eta^2} \frac{\partial}{\partial \eta} \left(\eta^2 \frac{\partial u_1^{(g)}}{\partial \eta} \right) - \left(\frac{\partial u_1^{(g)}}{\partial \eta} \right)^2 -$$
$$\frac{a_0^2}{\varkappa_g} \left[(\varphi + 2a_1) \frac{\partial u_1^{(g)}}{\partial t_0} - k_\gamma (u_1^{(g)} + 2a_1) \frac{d\varphi}{dt_0} \right] -$$
$$\left\{ \frac{1}{\eta^2} (j_1 + q_1^{(g)}) - \frac{a_0^2}{\varkappa_g} \left(\eta - \frac{1}{\eta^2} \right) \left(\frac{\partial a_1}{\partial t_0} + \frac{1}{3\gamma} \frac{d\varphi}{dt_0} \right) \right\} \frac{\partial u_1^{(g)}}{\partial \eta}$$
$$\beta_p = \frac{p_*}{\beta_*} \frac{\partial \beta}{\partial p} \Big|_{p_*, T_*}, \quad \beta_T = \frac{T_*}{\beta_*} \frac{\partial \beta}{\partial T} \Big|_{p_*, T_*}$$

where β_p and β_T are the dimensionless derivatives of the coefficient of accommodation with respect to pressure and temperature which characterize the "non-linear non-equilibrium nature" of the phase transition.

For the existence of t_0-periodic solutions of system (2.3) when $m = 2$, it is sufficient that the zeroth harmonics $f_{20}^{(\alpha)\circ}$ should satisfy condition (2.9). The zeroth harmonic of the product of two functions of the form

$$x^{(k)} = \operatorname{Re} \left\{ \sum_{n=1}^{\infty} x_n^{(k)\circ} e^{in\omega t_0} \right\}, \quad k = 1, 2$$

can be isolated by integrating this product over the period of the fluctuations.

$$(x^{(1)} x^{(2)})_0{}^\circ = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} x^{(1)} x^{(2)} \, dt_0 = \frac{1}{2} \sum_{n=1}^{\infty} \operatorname{Re} \{ \overline{x_n^{(1)\circ}} x_n^{(2)\circ} \}$$

Since the functions $f_2^{(\alpha)}$ are sums of products of this type then, while omitting the rather unwieldy reduction associated with evaluating the integrals with respect to t_0 and η, condition (2.9) can be written in the form of the second-order differential equation

$$\left(a_0 + \frac{k_s}{k_\gamma} d_\sigma \right) \frac{\partial a_0}{\partial t_2} = \frac{\varkappa_g k_\rho k_s}{2k_\gamma} \sum_{n=1}^{\infty} \left\{ - \left(\frac{1}{2k_s} + \beta_T - \frac{3}{2} \right) | \overset{\circ}{\theta}_{1n} |^2 + \right.$$
$$k_s \beta_p | \varphi_n{}^\circ |^2 + \operatorname{Re} \left\{ - \left[\beta_p - k_s \left(\beta_T - \frac{1}{2} \right) \right] \varphi_n{}^\circ \overline{\overset{\circ}{\theta}_{1n}} + (\overset{\circ}{\theta}_{1n} - k_\gamma \varphi_n{}^\circ) \times \right.$$
$$\left[\left(\frac{1}{2} + \sqrt{s_n} \right) \overline{\overset{\circ}{\theta}_{1n}} - \overline{\overset{\circ}{j}_{1n}} - \overline{q_{1n}^{(g)\circ}} - \left(\frac{k_\gamma}{2} - \frac{\sqrt{s_n}}{3\gamma} \right) \overline{\varphi_n{}^\circ} + \right.$$
$$\left. \left. \left. \left((1 - k_\rho) \overline{\overset{\circ}{j}_{1n}} + \overline{q_{1n}^{(g)\circ}} - \frac{s_n}{3\gamma} \overline{\varphi_n{}^\circ} \right) E_n \right] + k_\rho k_\lambda \left(\frac{1}{k_\varkappa} - \frac{1}{s_n} h_n^{(l)} \right) \overset{\circ}{\theta}_{1n} \overline{\overset{\circ}{j}_{1n}} \right\} \right\} \tag{4.2}$$
$$E_n = e^{\sqrt{s_n}} E_3 (\sqrt{s_n}), \quad E_3(z) = \int_1^{\infty} e^{-z\zeta} \zeta^{-3} \, d\zeta$$

(the bar indicates a complex conjugate). The magnitude of k_ρ in the factor $(1 - k_\rho)$ can be neglected compared with unity since the initial system (1.1) is valid in the domain of thermophysical parameters remote from the critical point, that is, when $k_\rho \ll 1$.

The results of calculations using Eq.(4.2) for a system with the thermophysical parameters of water and water vapour at a pressure $p_* = 0.1$ MPa are shown in Figs.2 and 3. On account of the lack of reliable data for the coefficient of accommodation which, moreover, is sensitive to small concentrations of impurities, surfactants and so on, the curves presented in the figures were calculated at β_* values equal to 4×10^{-4}, 4×10^{-2} and 4×10^{-1} (the solid curves) and using a quasi-equilibrium phase transition scheme (the broken curve). In the versions shown in the figures, the parameters characterizing the "non-linear non-equilibrium nature" β_p and β_T had the values $\beta_p = 0, \beta_T = 1.5$. This corresponds to the theoretical formula due to Landau /4/. It should be noted that the variation of β_T within reasonable limits had a weak effect on the results of the calculations. The frequency ν of the acoustic field, which was assumed to be monochromatic, was equal to 100 kHz for the curves with $\beta_* \neq \infty$. When the quasi-equilibrium phase transition scheme is used, the variables plotted along the coordinate axes are selfsimilar.

A phase portrait of the equation for the dynamics of the mean radius of the drop (4.2) is shown in Fig.2. It is seen that, with the value of the thermophysical parameters which were used, the rate of growth of the mean radius of the drop is negative, that is, the drop is evaporating on average. At large a_0, each of the phase trajectories tends to its own horizontal asymptote which is determined by the coefficient of accommodation β_* which suggests that a linear law exists for the reduction in the mean radii of the drops when their dimensions are sufficiently large or the frequencies are sufficiently high. At small values of a_0 and not too small values of β, the rate of decrease in the square of the mean radius (of the mean surface) of the drop is close to being constant.

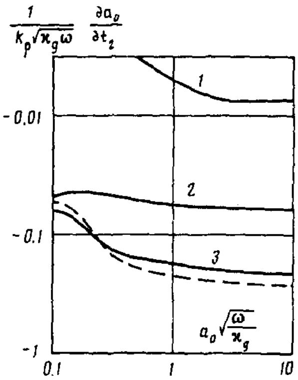

Fig.2

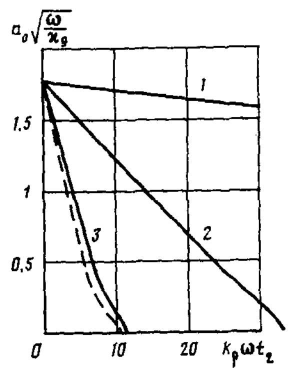

Fig.3

Fig.3 shows the dependence of the mean radius of the drop on the time t_2 obtained by numerical integration of Eq.(4.2). The shape and the mutual arrangement of the curves for different values of β_* depends, as follows from the phase portrait in Fig.2, from the choice of the initial condition $a_{00} = a_0(0)$ in the Cauchy problem for (4.2) (in the versions which have been presented, $a_{00} = 10$ μm). According to the calculations, the mean radius of the drop decreases to zero. In this connection, one should recall the constraint imposed on the mean radius which was stipulated above and, also, the fact that the given theory only considers those radii when a continuum description of the medium is permissible /1/. Nevertheless, one may speak of the lifetime of the drop which, judging from the calculations, increase appreciably as β_* decreases. Hence, in the case when $\beta_* = 0.04$ which is the value recommended for water /1, 4/, it may be several times greater than that calculated using the quasi-equilibrium phase transition scheme(this depends both on the frequency of the field and on the initial radius a_{00}).

It may be pointed out that the dynamics of the mean radius of a condensed drop in an acoustic field is determined according to (4.2), (3.4) and (3.5) by the eight independent dimensionless parameters $(\beta_*, \beta_p, \beta_T, k_s, k_\gamma, k_\rho, k_\lambda, k_\varkappa)$ plus the dependence on the initial radius, the frequency and the form of the fluctuations. This makes a complete analysis of Eq.(4.2) in the parameter space exceedingly difficult. It is therefore useful to consider certain characteristic limiting situations. In particular, simpler asymptotic equations can be obtained from (4.2), (3.4) and (3.5) which describe the behaviour of the mean radii of large and small drops in high- and low-frequency acoustic fields.

Let the frequencies ω or the radii a_0 be so large that the characteristic thickness of the non-stationary temperature boundary layer in the gas $\delta_g = [\varkappa_g/(2\omega)]^{1/2}$ is much smaller than the size of the drop $\delta_g \ll a_0$. In this case, the characteristic thickness of the analogous boundary layer in the liquid $\delta_l = [\varkappa_l/(2\omega)]^{1/2}$ will also be much smaller than a_0 since, as a rule, $\varkappa_l < \varkappa_g$ and even $\varkappa_l \ll \varkappa_g$. By using the asymptotic representations of the functions appearing in (4.2), (3.4) and (3.5) when the values of $|s_n|^{1/2}$ are large and solely retaining the leading terms in the expansions, one can obtain the equation (the high-frequency approximation)

$$\left(1 + \frac{k_s}{k_\gamma}\frac{d_\sigma}{a_0}\right)\frac{\partial a_0}{\partial t_2} = -\frac{1}{2}k_\rho k_s k_\gamma\sqrt{\varkappa_g}\sum_{n=1}^{\infty}|\varphi_n{}^\circ|^2\sqrt{\frac{\omega n}{2}}\,H(\sqrt{\omega_\sigma n}),\quad \omega_\sigma = \frac{d_\sigma{}^2}{\delta_g{}^2} \tag{4.3}$$

Here, $H(x) = P(x)/Q(x)$ is a rational fractional function, the coefficients of which are defined in terms of the thermophysical parameters of the phases.

$$P(x) = {}^1/_2 b_2 x^2 + b_1 x + b_0,\quad Q(x) = {}^1/_2 K^2 x + Kx + 1;\quad K = k_s(1 + k_\lambda/\sqrt{k_\varkappa})/k_\gamma$$

$$b_2 = K^2 - \frac{k_s}{k_\gamma} K, \quad b_1 = \frac{k_s}{k_\gamma} K^2 +$$
$$\left[\left(1 - \frac{k_s}{k_\gamma}\right)^2 - \frac{k_s^2}{k_\gamma^2}(2 - k_c)\right] K + \frac{k_s^2}{k_\gamma^2}(1 - k_c)$$
$$b_0 = \frac{k_s}{k_\gamma}\left[1 - \frac{k_s}{k_\gamma}(1 - k_c)\right] K + \left(1 - \frac{k_s}{k_\gamma}\right)^2 - k_c \frac{k_s^2}{k_\gamma^2}, \quad k_c = \frac{k_\rho k_\lambda}{k_\varkappa} = \frac{c_l}{c_g}$$

For many substances $K \gg 1$ and, therefore, usually only the leading terms are retained in the expressions for the coefficients $b_i\,(i = 0, 1, 2)$

Now let the function φ contain a finite number of harmonics ($\varphi_n^\circ = 0$ when $n > N$) and the frequencies and radii be so small that $\delta_g \gg a_0$, $\delta_l^2 \gtrsim a_0^2$, $(k_\lambda/120)(a_0/\delta_l)^4 \ll 1$ in the case of the highest frequency harmonic (relatively thick temperature boundary layers). By using the asymptotic representations of the functions appearing in (4.2), (3.4) and (3.5) for small value of $|s_n|^{1/2}$ and retaining the leading terms in the expansions (the magnitude of k_λ may be large, $k_\rho \ll 1$), (4.2) can be reduced to the form (the low-frequency approximation)

$$a_0\left(1 + \frac{k_s}{k_\gamma}\frac{d_\sigma}{a_0}\right)\frac{\partial a_0}{\partial t_2} = -\frac{\varkappa_g}{4}\frac{k_\rho k_s^2}{k_\gamma}\sum_{n=1}^{N}|\varphi_n^\circ|^2 G_n\left(\frac{k_s}{k_\gamma}\frac{d_\sigma}{a_0}\right) \tag{4.4}$$
$$G_n(x) = \frac{R_n(x)}{D_n(x)}, \quad R_n(x) = \alpha_{2n}x^2 + \alpha_{1n}x + \alpha_0,$$
$$D_n(x) = \left[1 + \left(\frac{k_\lambda |s_n|}{3k_\varkappa}\right)^2\right]x^2 + 2x + 1$$
$$\alpha_0 = 1 - k_\gamma - k_s\left[\frac{k_s}{k_\gamma} + \frac{4}{3}k_c\left(1 - \frac{k_s}{k_\gamma}\right)\right]$$
$$\alpha_{1n} = \frac{2k_\gamma}{k_s}\left\{1 - 2k_s + \left(1 - \frac{k_s}{k_\gamma}\right)\left[\beta_p + k_s\left(\beta_T - \frac{1}{2}\right)\right] + \right.$$
$$\left. k_s\left[\frac{k_s}{k_\gamma}\left(1 + \left(\frac{k_\lambda |s_n|}{3k_\varkappa}\right)^2\right) + \frac{4}{3}k_c\left(1 - \frac{k_s}{k_\gamma}\right)\right]\right\}$$
$$\alpha_{2n} = \frac{k_\gamma^2}{k_s^2}\left[1 - 2k_s - 2k_s\left(1 - \frac{k_s}{k_\gamma}\right)\left(\beta_T - \frac{1}{2}\right)\right] - 2\beta_p\left(1 - \frac{k_\gamma}{k_s} + \frac{k_\lambda^2 |s_n|^2}{9k_\varkappa^2}\right)$$

5. Discussion. The overall phase transition scheme can be thought of as consisting of two different mechanisms: a diffusion mechanism (resulting from the thermal conduction of the media) and a kinetic mechanism (associated with the deviation of the temperature of the surface of the drop from the saturation temperature). Needless to say, such a separation is arbitrary since the kinetic mechanism also involves molecular transport processes in the gas, that is, thermal conduction. The quasi-equilibrium phase transition scheme is a scheme in which there is only a diffusion mechanism. It is possible to introduce the concept of a zone where the kinetic mechanism is operative (also, see /5/) and the characteristic width of this zone in the gas δ_σ is, by definition, equal to

$$\delta_\sigma = \frac{k_s}{k_\gamma} d_\sigma = \frac{1 - k_\rho}{\gamma - 1}\sqrt{\frac{2\pi}{\gamma}}\frac{1}{\beta_*}\frac{\varkappa_g C_g^3}{l^2}$$

Here, the explicit expression for k_s has been taken from the Clausius-Clapeyron equation /1/, $C_g = (\gamma p_*/\rho_{g*})^{1/2}$ is the velocity of sound in the gas and the thermal conductivity of the gas can be eliminated from the relationship since $\varkappa_g \sim L_f C_g$, where L_f is the mean free path of the molecules in the gas.

If $\delta_\sigma \gtrsim a_0$, the non-equilibrium phase transition kinetics have an effect on the quasi-stationary temperature distributions. This situation is only possible in the case of extremely small value of β or a small value of a_0 and is atypical. Usually, $\delta_\sigma \ll a_0$ and the corresponding terms in (4.2)-(4.4) can be neglected. Then, as can be seen from (4.3), the mean radius of a drop in a high-frequency field varies linearly.

The non-equilibrium phase transition kinetics have an effect on the behaviour of the drops in high-frequency fields when $\delta_g \ll a_0$, $\delta_l \ll a_0$ and the magnitude of δ_σ is comparable with δ_g or δ_l. Hence, estimates according to Eq.(4.3) for water when $p_* = 0.1$ MPa and $\beta_* = 0{,}04$ show that the quasi-equilibrium scheme ($H(\sqrt{\omega_\sigma}) \approx H(0) = b_0$) can only be used at frequencies $\nu = \omega/2\pi \lesssim 0.1$ kHz. Both mechanisms manifest themselves in the range 0.1 kHz $\ll \nu \ll 1.0$ MHz while, in the range $\nu \gtrsim 1.0$ MHz, the kinetic mechanism plays the predominant role. Several estimates and illustrations of the influence of the non-equilibrium nature of the phase transition on the dispersion and attenuation of sound in a mixture of vapour and drops have been given in /3/.

At low frequencies ($(k_\lambda/6)^2(a_0/\delta_l)^4 \ll 1$) and, also, in the case of a quasi-equilibrium phase transition $(d_\sigma = 0)$, the right-hand side of (4.4) is independent of the frequency, and the behaviour of the mean radius is determined by the mean square value of the function φ $(\langle\varphi^2\rangle = {}^1/_2\,(|\varphi_1^\circ|^2 + \ldots + |\varphi_N^\circ|^2))$. From (4.4), it is possible to find the lifetime of a drop (t_*) in the case of the quasi-equilibrium phase transition scheme since, when $d_\sigma = 0$, the equation is integrated and

$$a_0 = \left[a_{00}^2 - \alpha_0 \frac{k_\rho k_s^2}{k_\gamma} \langle\varphi^2\rangle \varkappa_g t_2\right]^{1/2}; \quad t_* = \frac{k_\gamma}{\alpha_0 k_\rho k_s^2 \varepsilon^2 \langle\varphi^2\rangle} \frac{a_{00}^2}{\varkappa_g} \tag{5.1}$$

In spite of the fact that, according to the calculations which have been carried out, condensed drops under normal conditions usually evaporate under the action of acoustic fields, one should be warned against prematurely arriving at the conclusion that this phenomenon will be observed in the case of the drops of other substances or drops of water under other conitions. Domains exist in the parameter space where the drops will grow (for example, the coefficient α_0 in (4.4) and (5.1) is negative), and situations are possible when stable and unstable zeros (stationary and threshold radii) arise in the phase portrait of Eq.(4.2) or (4.4). The detailed analysis of the phase portrait of the equation for the dynamics of the mean radius of a drop and also the physical realizability of "anomalous" parameter domains is obviously of interest in its own right.

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